

Enstrophy dissipation in freely evolving two-dimensional turbulence

Chuong V. Tran*

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

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Abstract

Freely decaying two-dimensional Navier–Stokes turbulence is studied. The conservation of vorticity by advective nonlinearities renders a class of Casimirs that decays under viscous effects. A rigorous constraint on the palinstrophy production by nonlinear transfer is derived, and an upper bound for the enstrophy dissipation is obtained. This bound depends only on the decaying Casimirs, thus allowing the enstrophy dissipation to be bounded from above in terms of initial data of the flows. An upper bound for the enstrophy dissipation wavenumber is derived and the new result is compared with the classical dissipation wavenumber.

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In 1969, Batchelor¹ adapted Kolmogorov's equilibrium theory for three-dimensional (3D) turbulence to two-dimensional (2D) turbulence, on the basis of a phenomenologically analogous property between the two systems. For a 3D fluid, decrease of the viscosity ν is accompanied by increase of the mean-square vorticity, a consequence of the magnification of the vorticity by stretching of vortex lines, so that in the inviscid limit the energy dissipation is nonzero. For a 2D fluid, decrease of the viscosity enhances convective mixing, in which isovorticity lines get extended and brought closer to one another,¹ giving rise to increase of the mean-square vorticity gradients (twice the palinstrophy), so that in the inviscid limit the rate of enstrophy (half the mean-square vorticity) dissipation can approach a finite value χ . On the basis of this analogy, Batchelor¹ applies the familiar arguments of the Kolmogorov equilibrium theory for the small-scale components of 3D turbulence to the 2D case, where the roles of the energy and energy dissipation in the original theory are played by the enstrophy and enstrophy dissipation. This means that the statistical properties of the small-scale components of the turbulence depend only on the two dimensional parameters χ and ν . The enstrophy dissipation χ is thus an important dynamical quantity in Batchelor's theory. One of its prominent role is in the expression of the enstrophy spectrum $Z(k)$ of the so-called enstrophy inertial range, which is presumably formed when an initial enstrophy reservoir spreads out in a virtually inviscid region of wavenumber space:

$$Z(k) = C\chi^{2/3}k^{-1}, \quad (1)$$

where C is a universal constant and k is the wavenumber. Another important role of χ is in the determination of the dissipation wavenumber k_ν :

$$k_\nu = \frac{\chi^{1/6}}{\nu^{1/2}}, \quad (2)$$

which presumably marks the end of the enstrophy inertial range, around which the enstrophy is most strongly dissipated. In both (1) and (2), χ is a finite (but otherwise undetermined) parameter.

It is desirable to have a quantitative knowledge of χ (and hence of k_ν), not only for its role in the Batchelor theory but also for further analyses of the turbulence, beyond the usual dimensional arguments. Even for finite Reynolds numbers, the determination of χ is highly non-trivial. In the limit $\nu \rightarrow 0$, this problem does not seem to become more tractable. Ideally, χ can be determined if the extent to which the production of

palinstrophy by convective mixing is fully understood. This letter takes a direct approach to this problem by deriving a rigorous upper bound for the nonlinear term representing the palinstrophy production rate. Equating this bound to the viscous dissipation term yields a constraint, from which upper bounds for χ and for the enstrophy dissipation wavenumber k_d (to be defined later in this letter) can be derived. These bounds are found to be completely described in terms of initial data of the flows. The derived enstrophy dissipation wavenumber is consistent with the classical prediction of k_ν given by (2), in the sense that they both have the same functional dependence on ν . A novel result of this study is that χ and k_d can be estimated in terms of initial data of the turbulence, while they (more precisely χ and k_ν) are essentially undetermined in the Batchelor theory.

In the vorticity formulation, the freely evolving 2D Navier–Stokes equations governing the motion of an incompressible fluid confined to a doubly periodic domain are

$$\partial_t \xi + J(\psi, \xi) = \nu \Delta \xi, \quad (3)$$

where $\xi(\mathbf{x}, t)$ is the vorticity, $J(\theta, \vartheta) = \theta_x \vartheta_y - \theta_y \vartheta_x$, ν the kinematic viscosity, and $\psi(\mathbf{x}, t)$ the stream function. The vorticity is defined in terms of the stream function and of the velocity \mathbf{v} by $\xi = \Delta \psi = \hat{n} \cdot \nabla \times \mathbf{v}$, where \hat{n} is the normal vector to the fluid domain. Equivalently, \mathbf{v} can be recovered from ψ and ξ by $\mathbf{v} = (-\psi_y, \psi_x) = (-\Delta^{-1} \xi_y, \Delta^{-1} \xi_x)$.

An importance property of the advective nonlinear term in (3) is that it conserves the kinetic energy and an infinite class of integrated quantities, known as Casimirs, including the enstrophy. The latter conservation law is attributed to the fact that vorticity is conservatively redistributed in physical space by the advective transfer. While the conservation of energy and enstrophy imposes strict constraints on turbulent flows and has been explored in the literature to a great extent,^{2–19} the conservation of Casimirs, other than the enstrophy, seems to render little additional knowledge of the flows and has received much less attention.^{20–22} In the present case of unforced dynamics, it is well known that a wide class of these Casimirs decays under the action of viscosity. Here our main interest is in the following Casimirs: $\langle |\xi|^p \rangle$, where $\langle \cdot \rangle$ denotes a spatial average. By taking the time derivative of $\langle |\xi|^p \rangle$ and using (3) one obtains

$$\begin{aligned} \frac{d}{dt} \langle |\xi|^p \rangle &= -p \langle |\xi|^{p-2} \xi J(\psi, \xi) \rangle + \nu p \langle |\xi|^{p-2} \xi \Delta \xi \rangle \\ &= -\nu p(p-1) \langle |\xi|^{p-2} |\nabla \xi|^2 \rangle, \end{aligned} \quad (4)$$

where the second equation is obtained by integration by parts, upon which the nonlinear term identically vanishes. For $p = 2$, Eq. (4) governs the decay of $\langle |\xi|^2 \rangle$ (twice the enstrophy), for which the dissipation term becomes 2χ , which is the subject of this study. The right-hand side of (4) is negative for $p > 1$. Hence $\langle |\xi|^p \rangle$ decays in time, for $p > 1$. It follows that

$$\langle |\xi(t)|^p \rangle \leq \langle |\xi(0)|^p \rangle, \quad (5)$$

for $p > 1$ and $t \geq 0$. In particular, in the limiting case $p \rightarrow \infty$, one has

$$\|\xi(t)\|_\infty \leq \|\xi(0)\|_\infty, \quad (6)$$

for $t \geq 0$, where $\|\xi\|_\infty$ denotes the L^∞ norm of ξ .

Now the main result of this letter can be readily derived. By multiplying (3) by $\Delta\xi$ and taking the spatial average of the resulting equation, one obtains the equation governing the evolution of the palinstrophy $\langle |\nabla\xi|^2 \rangle/2$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle |\nabla\xi|^2 \rangle &= \langle \Delta\xi J(\psi, \xi) \rangle - \nu \langle |\Delta\xi|^2 \rangle \\ &= \langle \xi J(\psi_x, \xi_x) \rangle + \langle \xi J(\psi_y, \xi_y) \rangle - \nu \langle |\Delta\xi|^2 \rangle \\ &\leq \langle |\xi| (|\nabla\psi_x| |\nabla\xi_x| + |\nabla\psi_y| |\nabla\xi_y|) \rangle - \nu \langle |\Delta\xi|^2 \rangle \\ &\leq \|\xi\|_\infty \langle |\nabla\psi_x|^2 + |\nabla\psi_y|^2 \rangle^{1/2} \langle |\nabla\xi_x|^2 + |\nabla\xi_y|^2 \rangle^{1/2} - \nu \langle |\Delta\xi|^2 \rangle \\ &\leq \|\xi\|_\infty \langle |\xi|^2 \rangle^{1/2} \langle |\Delta\xi|^2 \rangle^{1/2} - \nu \langle |\Delta\xi|^2 \rangle. \end{aligned} \quad (7)$$

In (7), the second equation is obtained via the two elementary identities

$$\Delta J(\psi, \xi) = J(\psi, \Delta\xi) + 2J(\psi_x, \xi_x) + 2J(\psi_y, \xi_y) \quad (8)$$

and

$$\langle \Delta\xi J(\psi, \xi) \rangle = -\langle \xi J(\psi, \Delta\xi) \rangle. \quad (9)$$

The Hölder inequality is used in the fourth step, and the last step can be seen by expressing ψ (and ξ) in terms of Fourier series. The triple-product term $\|\xi\|_\infty \langle |\xi|^2 \rangle^{1/2} \langle |\Delta\xi|^2 \rangle^{1/2}$ represents an upper bound for the palinstrophy production rate. It can be seen that the palinstrophy necessarily ceases to grow as its dissipation $\nu \langle |\Delta\xi|^2 \rangle$ reaches this bound. It follows that as the palinstrophy grows to and reaches a maximum ($d\langle |\nabla\xi|^2 \rangle/dt \geq 0$), the following inequality necessarily holds

$$\langle |\Delta\xi|^2 \rangle^{1/2} \leq \frac{\|\xi\|_\infty \langle |\xi|^2 \rangle^{1/2}}{\nu}. \quad (10)$$

Since $\langle |\nabla \xi|^2 \rangle$ can be bounded from above in terms of $\langle |\Delta \xi|^2 \rangle$, ineq. (10) can be used to derive an explicit upper bound for the palinstrophy. By Hölder inequality one has¹⁸

$$\langle |\Delta \xi|^2 \rangle \geq \frac{\langle |\nabla \xi|^2 \rangle^2}{\langle |\xi|^2 \rangle}, \quad (11)$$

where the inequality sign “ \geq ” can become “ \gg ” (see below). Substituting (11) into (10) yields

$$\langle |\nabla \xi|^2 \rangle \leq \frac{\|\xi\|_\infty \langle |\xi|^2 \rangle}{\nu}. \quad (12)$$

It follows that the enstrophy dissipation χ is bounded from above by

$$\chi \leq \|\xi\|_\infty \langle |\xi|^2 \rangle. \quad (13)$$

It is notable that the upper bound for χ in (13) is expressible in terms of two decaying Casimirs, namely $\|\xi\|_\infty$ and $\langle |\xi|^2 \rangle$, so that it can be bounded from above in terms of initial data of the flows. More accurately, χ can be bounded from above in terms of the initial vorticity field only. In passing, it is worth mentioning that since $\|\xi\|_\infty$ and $\langle |\xi|^2 \rangle$ are intensive quantities, i.e. independent of the domain size, the constraint (13) is size-independent.

Let us denote by k_d and k_D the wavenumbers defined by $\langle |\nabla \xi|^2 \rangle^{1/2} / \langle |\xi|^2 \rangle^{1/2}$ and $\langle |\Delta \xi|^2 \rangle^{1/2} / \langle |\nabla \xi|^2 \rangle^{1/2}$, respectively. For “regular” spectra, k_d (k_D) specifies where, in wavenumber space, $\langle |\nabla \xi|^2 \rangle$ ($\langle |\Delta \xi|^2 \rangle$) is mainly distributed. In other words, k_d (k_D) specifies where, in wavenumber space, the enstrophy (palinstrophy) dissipation mainly occurs. By “regular” it is meant that the enstrophy is not highly concentrated in any particular regions of wavenumber space that would result in severe steps in the enstrophy spectrum. By (11), these dissipation wavenumbers satisfy $k_d \leq k_D$, and the inequality sign “ \leq ” can become “ \ll ”. This can be realized if the palinstrophy spectrum around k_d does not fall off so steeply. More quantitatively, by the definition of k_d , one can expect the palinstrophy spectrum around k_d to be shallower than k^{-1} . (Because, otherwise, most of the contribution to $\langle |\nabla \xi|^2 \rangle$ would come from $k < k_d$, making the ratio $\langle |\nabla \xi|^2 \rangle^{1/2} / \langle |\xi|^2 \rangle^{1/2}$ significantly lower than k_d , a contradiction to the very definition of k_d .) This means that the spectrum of $\langle |\Delta \xi|^2 \rangle$ around k_d is shallower than k^1 . Hence, most of the contribution to $\langle |\Delta \xi|^2 \rangle$ can come from $k \gg k_d$ if the palinstrophy spectrum beyond k_d becomes steeper than k^{-1} and falls off to k^{-3} gradually. This allows for the possibility $k_D \gg k_d$ to be realized. In any case, k_d should be well beyond the end of the enstrophy range, i.e. $\int_{k > k_d} Z(k) dk / \int_{k < k_d} Z(k) dk \approx 0$, and k_D

should be well beyond the palinstrophy range, i.e. $\int_{k>k_D} P(k) dk / \int_{k<k_D} P(k) dk \approx 0$, where $Z(k)$ and $P(k)$ are the enstrophy and palinstrophy spectra, respectively. In other words, the enstrophy spectrum around k_d should be steeper than k^{-1} , and the palinstrophy (enstrophy) spectrum around k_D should be steeper than k^{-1} (k^{-3}).

Our primary concern is an estimate of k_d when $\langle |\nabla \xi|^2 \rangle$ achieves a maximum. It is likely that k_d achieves a global maximum then. Equation (7) and the subsequent equations (10) and (12) imply

$$k_d k_D \leq \frac{\|\xi\|_\infty}{\nu} \quad (14)$$

and

$$k_d \leq \left(\frac{\|\xi\|_\infty}{\nu} \right)^{1/2} \leq \left(\frac{\|\xi(0)\|_\infty}{\nu} \right)^{1/2}. \quad (15)$$

It is interesting to compare the present result with the classical dissipation wavenumber k_ν . To this end, let us express $\chi = \nu \langle |\nabla \xi|^2 \rangle$ in the form $\chi = \nu k_d^2 \langle |\xi|^2 \rangle$, so that (2) can be rewritten as

$$k_\nu = \frac{\chi^{1/6}}{\nu^{1/2}} = \left(\frac{k_d \langle |\xi|^2 \rangle^{1/2}}{\nu} \right)^{1/3}. \quad (16)$$

It follows that

$$\frac{k_\nu^3}{k_d} = \frac{\langle |\xi|^2 \rangle^{1/2}}{\nu}. \quad (17)$$

It may be assumed that $\|\xi\|_\infty$ and $\langle |\xi|^2 \rangle^{1/2}$ are of the same order of magnitude. In fact, one can even have the exact equality $\|\xi\|_\infty = \langle |\xi|^2 \rangle^{1/2}$ for some simple cases. For example, for a vorticity field of a single Fourier mode, this equality trivially holds. In any case, both $\|\xi\|_\infty$ and $\langle |\xi|^2 \rangle^{1/2}$ decay under the action of viscosity, so that if they are comparable initially, then one can expect them to be comparable subsequently. Hence one can deduce from (14) and (17) that

$$k_d^2 k_D \approx k_\nu^3. \quad (18)$$

It follows that $k_d \leq k_\nu$, and k_d can be significantly lower than k_ν if $k_d \ll k_D$.

As pointed out by a referee, the result (15) can be derived from more physical considerations, based on the relation

$$\partial_t \langle \xi(1) \xi(2) \rangle + \partial_r \langle \delta v \xi(1) \xi(2) \rangle \approx \nu \partial_r^2 \langle \xi(1) \xi(2) \rangle, \quad (19)$$

where 1 and 2 stand for \mathbf{x}_1 and \mathbf{x}_2 , respectively, $r = |\mathbf{x}_2 - \mathbf{x}_1|$, and δv denotes the fluid speed increment across r . In the limit $r \rightarrow k_d^{-1}$, if one assumes $\xi \approx \delta v / r$ then one can recover (15) by balancing the nonlinear and viscous terms in (19).

In conclusion, this letter has derived a rigorous constraint on the palinstrophy production rate by nonlinear transfer, from which upper bounds for the enstrophy dissipation χ and for the enstrophy dissipation wavenumber k_d have been deduced. These bounds are expressible in terms of the vorticity supremum $\|\xi\|_\infty$, the mean-square vorticity density $\langle |\xi|^2 \rangle$, and the viscosity ν . These quantities are “intensive” quantities, i.e. independent of the domain size, making the derived upper bounds size-independent. Moreover, these bounds are completely determined by the initial vorticity field and ν since both $\|\xi\|_\infty$ and $\langle |\xi|^2 \rangle$ decay under the action of viscosity. The upper bound $k_d \leq \|\xi(t)\|_\infty^{1/2} / \nu^{1/2} \leq \|\xi(0)\|_\infty^{1/2} / \nu^{1/2}$ is consistent with the classical dissipation wavenumber $k_\nu = \chi^{1/6} / \nu^{1/2}$, in the sense that they both have the same functional dependence on ν . A novel result of this study is that both χ and k_d are bounded in terms of the initial vorticity field and the viscosity whereas χ and k_ν are essentially undetermined by the classical theory.

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¹G. K. Batchelor, “Computation of the energy spectrum in homogeneous two-dimensional turbulence,” *Phys. Fluids* **12**, 233 (1969).

²G. Boffetta, A. Celani, and M. Vergassola, “Inverse energy cascade in two-dimensional turbulence: Deviations from Gaussian behaviour,” *Phys. Rev. E* **61**, R29 (2000).

³V. Borue, “Inverse energy cascade in stationary two-dimensional homogeneous turbulence,” *Phys. Rev. Lett.* **72**, 1475 (1994); “Spectral exponents of enstrophy cascade in stationary two-dimensional homogeneous turbulence,” *Phys. Rev. Lett.* **71**, 3967 (1993).

⁴S. Chen *et al.*, “Physical mechanism of the two-dimensional enstrophy cascade,” *Phys. Rev. Lett.* **91**, 214501 (2003).

⁵P. Constantin, C. Foias, and O. P. Manley, “Effects of the forcing function spectrum on the energy spectrum in 2-D turbulence,” *Phys. Fluids* **6**, 427 (1994).

⁶B. Dubrulle and S. Nazarenko, “Interaction of turbulence and large-scale vortices in incompressible 2D fluids,” *Physica D* **110**, 123 (1997).

- ⁷G. Eyink, “Exact results on stationary turbulence in 2d: consequences of vorticity conservation,” *Physica D* **91**, 97 (1996).
- ⁸R. Fjørtoft, “On the changes in the spectral distribution of kinetic energy for twodimensional, nondivergent flow,” *Tellus* **5**, 225 (1953).
- ⁹U. Frisch and P. L. Sulem, “Numerical simulation of the inverse cascade in two-dimensional turbulence,” *Phys. Fluids* **27**, 1921 (1984).
- ¹⁰R. H. Kraichnan, “Inertial ranges in two-dimensional turbulence,” *Phys. Fluids* **10**, 1417 (1967); “Inertial-range transfer in two- and three-dimensional turbulence,” *J. Fluid Mech.* **47**, 525 (1971).
- ¹¹S. B. Kuksin, “The Eulerian limit for 2D statistical hydrodynamics,” *J. Stat. Phys.* **115**, 469 (2004).
- ¹²C. E. Leith, “Diffusion approximation for two-dimensional turbulence,” *Phys. Fluids* **11**, 671 (1968).
- ¹³E. Lindborg and K. Alvelius, “The kinetic energy spectrum of the two-dimensional enstrophy turbulence cascade,” *Phys. Fluids* **12**, 945 (2000).
- ¹⁴P. E. Merilees and H. Warn, “On energy and enstrophy exchanges in two-dimensional non-divergent flow,” *J. Fluid Mech.* **69**, 625 (1975).
- ¹⁵J. Paret, M.-C. Jullien, and P. Tabeling, “Vorticity Statistics in the two-dimensional enstrophy cascade,” *Phys. Rev. Lett.* **83**, 3418 (1999).
- ¹⁶L. M. Smith, V. Yakhot, “Bose condensation and small-scale structure generation in a random force driven 2D turbulence,” *Phys. Rev. Lett.* **71**, 352 (1993); “Finite-size effects in forced, two-dimensional turbulence,” *J. Fluid Mech.* **271**, 115 (1994).
- ¹⁷C. V. Tran and J. C. Bowman, “Robustness of the inverse cascade in two-dimensional turbulence,” *Phys. Rev. E* **69**, 036303 (2004); “On the dual cascade in two-dimensional turbulence,” *Physica D* **176**, 242 (2003).
- ¹⁸C. V. Tran, “Nonlinear transfer and spectral distribution of energy in α turbulence,” *Physica D* **191**, 137 (2004).
- ¹⁹C. V. Tran and T. G. Shepherd, “Constraints on the spectral distribution of energy and enstrophy dissipation in forced two-dimensional turbulence,” *Physica D* **165**, 201 (2002).
- ²⁰D. D. Holm *et al.*, “Nonlinear stability of fluid and plasma equilibria,” *Phys. Rep.* **123**, 1 (1985).
- ²¹Y. C. Li, “On 2D Euler equations. I. On the energy-Casimir stabilities and the spectra

for linearized 2D Euler equations,” J. Math. Phys. **41**, 728 (2000).

²²D. Wirosoetisno and T. G. Shepherd, “Nonlinear stability of Euler flows in two-dimensional periodic domains,” Geophys. Astrophys. Fluid Dynamics **90**, 229 (1999).

* Electronic address: ctran@maths.warwick.ac.uk